Vector lattice model for stresses in granular materials

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A vector lattice model for stresses in granular materials is proposed. A two-dimensional pile built by pouring from a point is constructed numerically according to this model. Remarkably, the pile violates the Mohr Coulomb stability criterion for granular matter, probably because of the inherent anisotropy of such poured piles. The numerical results are also compared to the earlier continuum fixed principal axes model and the (scalar) lattice q-model.

Understanding how granular materials sustain externally applied loads is an interesting and difficult scientific problem, with obvious engineering implications. Unlike for elastic materials, the slippages that occur when a granular heap is built make it impossible to define a displacement (and thence a strain) field. One thus has to solve for a tensor stress field (which cannot be expressed in terms of a vector displacement field) using the vector force balance equation. In two dimensions, where the stress tensor has three independent components, this results in one ''missing'' equation. In three dimensions, the situation is even worse.

One of the main tools used to deal with this difficulty is the Mohr Coulomb (MC) stability criterion. This states that the ratio of the shear to the normal stress at any point inside a cohesionless granular heap, with any orientation of axes, cannot exceed the coefficient of friction μ. This is natural, since otherwise one would expect an appropriately located (and oriented) slip plane to spontaneously destabilize the heap. Using the MC inequality and symmetry (or further assumptions), it is sometimes possible to obtain useful results on the stability of granular heaps [1]. For instance, the elastoplastic model [2] of granular stresses relies crucially on this criterion. The fixed principal axes (FPA) model [3,4] does not invoke the MC criterion, but obtains it as a byproduct.

Both the elastoplastic and the FPA theory yield the surprising result that the MC criterion is saturated in a large outer region of a two-dimensional pile built by pouring from a point. This implies [5] that an infinitesimal perturbation inside the outer region should destabilize the pile, while one would expect that a poured pile should only be unstable at the surface.

A byproduct of the MC criterion is that the angle of repose ϕ should satisfy the equation

$$\tan \phi = \mu. \tag{1}$$

In two dimensions, the MC criterion becomes

$$\frac{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2}{(\sigma_{xx} + \sigma_{yy})^2} \leq \sin^2 \phi. \tag{2}$$

Although the MC criterion seems reasonable, there is the implicit assumption (in the orientation-independent μ) that the granular medium is isotropic. This is a questionable as-

sumption for a granular heap built by pouring from a point. As the grains roll down the slopes of the heap, anisotropy can be ''frozen in,'' with different behavior parallel and perpendicular to the slope. In fact, for a two-dimensional heap of identical disks in a perfect triangular lattice [6], the left-hand side of Eq. (2) goes to unity at the corner of an infinitely high heap, while $\phi = \pi/3$. The motivation for this paper is to investigate whether this anisotropy persists away from the special case of the perfect lattice.

In this Rapid Communication, we consider a two-dimensional lattice model, with the grains placed on a triangular lattice (see Fig. 1). Each grain rests on its two neighbors in the row below, and supports its neighbors in the row above. Adjoining sites in the same horizontal row are not connected. Forces propagate downwards from each grain to its descendants. In addition to the force from its predecessors, a grain also transmits its own weight downwards. This weight is taken to be the same for every grain.

In order to include disorder in the model, the outgoing bond angles θ_1 and θ_2 from a site are chosen randomly. For N lattice sites, there are $2N$ random angles, each of which is chosen independently from a uniform distribution over the interval $[\theta_{\min}, \theta_{\max}]$. The width of this interval is a measure of the degree of disorder. With equal sized grains, even if the lattice is viewed as a topological rather than a geometrical structure, the bond angles would not be all independent, but such correlations are ignored for simplicity.

Vector force balance is enforced on every lattice site. Solving for the intergrain force network, proceeding downwards from row to row, inevitably leads to occasional negative (i.e., attractive) intergrain forces. This is unphysical for a
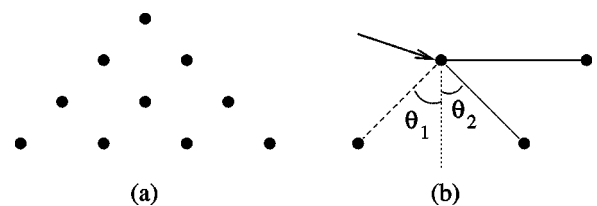

FIG. 1. (a) Triangular lattice. Each site rests on its two neighbors below, and receives forces from its neighbors above. (b) The incoming force is too horizontal, and the site is unstable. The downwards bond going left is broken, and a horizontal bond is established. The angles θ_1 and θ_2 are independent random variables at each site.

cohesionless granular medium. Accordingly, if the force between a grain and one of its descendants is negative, we interpret this to mean that the grain is unstable, and rolls in the direction of the other descendant. Within the model, this is accommodated by breaking the bond with negative force, and establishing a new bond with the adjacent site (in the same row as the unstable grain) in the opposite direction (see Fig. 1). The new bond established necessitates recomputing force balance for the sites in the row; several iterations can be required before a stable configuration is achieved.

In reality, a moved grain would also change the angles for its unbroken bonds. Moreover, since the pile is not really a regular lattice, the link established to the adjacent site would not be horizontal. Both of these issues are ignored. The idea of unstable grains readjusting contacts was suggested earlier [7], but the implementation of this process and the specific system studied are different here.

The model described above is similar to the earlier q-model [8,9] with one key difference: the q-model has *scalar* forces, only keeping track of the vertical force on each grain. Even with this limitation, the q-model successfully accounts for the experimental result [8,10,11] that the vertical forces on individual grains at the bottom of a granular heap have a distribution $P(f)$ that decays exponentially for large f. It also agrees with the experimental observation [11] that the vertical forces f_i and f_j on two grains at the bottom of a heap satisfy $\langle f_i f_j \rangle = \langle f_i \rangle \langle f_j \rangle$. However, the absence of vector forces in the q-model prevents it from reproducing the visually most striking feature of granular forces: the existence of ''force chains'' [12,8], a sparse network of grains that experience large stresses. Further, while forces on a single grain level are important for the failure of granular materials, one is often interested in a more coarse grained continuum description. There has been earlier work on vectorizing the q-model indirectly [13], and directly with a method different from the one here [14]. We shall see that the vector lattice model proposed in this paper agrees well with the same experimental results as the q-model, and in addition yields force chains, corroborates some aspects of the FPA model while disagreeing with others, and permits a test of the MC stability criterion.

The specific system we consider is a granular pile built by pouring from a point. This is the system on which most of the FPA model analysis has concentrated; also, we have reasons to question the validity of the MC criterion in such a system. For convenience, in the numerical simulations, the pile is built from the top down instead of from the bottom up. (No significant difference is found with a bottom up construction.) Thus, one starts with one lattice site in the top row and computes the outgoing forces, proceeding to two sites in the next row and so on. If a row has m sites, the next row has $m+1$ sites *unless* the sites at the edge require adjacent sites to establish horizontal bonds with. All the numerics in this paper are with bond angles ranging over $[\pi/6, \pi/3]$, unless specified otherwise.

Figure 2 shows the result of such a simulation, with 400 rows. There is a well defined slope to the pile. We view the central m sites in the mth row as the *interior* of the pile, and the sites that flank this inner region as *buttresses*, necessary
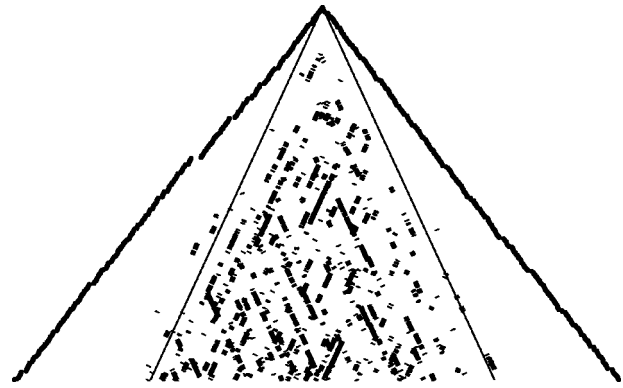

FIG. 2. A pile 400 rows high. The outer lines are the boundaries of the pile, and show that it has a well defined slope. In any row, bonds with a force greater than four times the average bond force *in that row* are marked, and show directed force chains. The thin straight lines are the boundaries of the inner region of the pile. The force chains are concentrated in the inner region. (If the threshold were less than 4, there would be more chains in the outer region.)

to stabilize the pile. This division into an inner and outer region is in accordance with the FPA model. With the high stress bonds marked, one sees force chains concentrated in the inner region of the pile, in accordance with experiment [15]. The force chains seem perfectly straight because the sites are on a regular lattice; with a more realistic model, where the random bond angles would be accompanied by an irregular lattice, the chains should meander, with a mean orientation of $\langle \theta \rangle$.

If the horizontal and vertical forces across the bottom of the pile (averaged over many runs) are plotted, the vertical force f_y has a flat peak in the inner region and falls off steeply in the outer region. The variation in f_y across the inner region is less than $\pm 5\%$. The horizontal force f_x varies linearly across the inner region, peaks roughly at the boundary between the inner and outer regions, and falls off in the outer region. The linear profile for f_x shows that the pile has a strong tendency to *splay*.

One can coarse grain and compute ''continuum'' stresses: Fig. 3 shows the components of the stress tensor, calculated by coarse graining over 20 lattice sites at the bottom of a pile of 8000 rows, and then averaging over 2000 runs. Not surprisingly, the division between the inner and outer regions seen in the force components f_x and f_y is also seen here. For piles more than 200 rows high (smaller piles have not been checked), stresses and forces scale linearly with the pile height.

The slight curvature seen in σ_{yy} in the central region is accentuated in the plot of σ_{xx}/σ_{yy}. The steady increase in this stress ratio as one moves away from the symmetry axis violates the hypothesis of Ref. [3], that $\sigma_{yy} \propto \sigma_{xx}$; this cannot be cured by using the more general FPA or oriented stress linearity (OSL) hypotheses [4] of $\sigma_{xx} = k_1 \sigma_{yy} + k_2 |\sigma_{xy}|$, since σ_{xx}/σ_{yy} is quadratic at the center of the pile, whereas $|\sigma_{xy}|$ has a linear kink. More importantly, the violation of the FPA hypothesis can be understood *physically*. At least in the inner region, the splay in the pile makes it increasingly likely for sites to be destabilized outwards as one moves away from
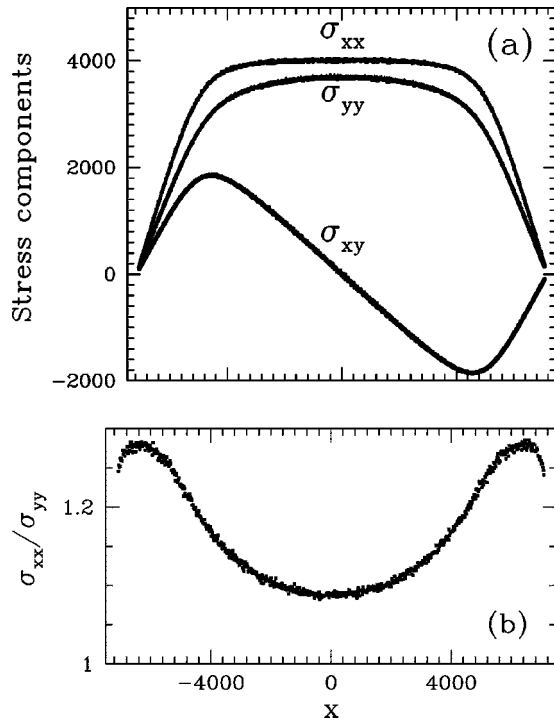
FIG. 3. (a) Components of the stress tensor across the bottom of the pile. The pile is 8000 rows high, so the boundary of the inner region is at $x = \pm 4000$. (x is in units of grain number, and the stress is in units of grain weight per lattice spacing.) (b) The ratio σ_{xx}/σ_{yy} from the same data, showing that σ_{yy} is not proportional to σ_{xx}. σ_{xx} cannot be expressed even as a linear combination of σ_{yy} and $|\sigma_{xy}|$.

the axis of symmetry. Such destabilized sites establish bonds with their adjacent sites. Thus, while the forces emerging from a site in the central region propagate downwards to its descendants, there is a steady drift towards more horizontal force propagation as one moves outwards. This reorients the principal axes of the stress tensor and increases σ_{xx}/σ_{yy}. Changing the range of bond angles, $[\theta_{\min}, \theta_{\max}]$ has no qualitative effect on the results discussed so far: the size of the buttresses flanking the inner region naturally increases when the bond angle range is increased.

The approximately flat profile of the vertical force in the inner region allows us to compute the distribution of vertical forces in this region. Figure 3 was obtained by averaging over many runs and lattice sites, but the forces on a single site in a single run have large fluctuations. Figure 4 shows a semilog plot of the probability $P(f)$ for a grain in (the inner region of) the bottom row to experience a force f. Although there is a slight deviation from an exponential for large f, this steadily decreases as the size of the pile is increased, suggesting an exponential tail to $P(f)$ for infinite system size. If the range of bond angles is reduced, the evolution towards an exponential tail for $P(f)$ with increasing system size is slower, but is manifest. Conversely, the inset to Fig. 4 is for a bond angle range of $[0.01\pi/3, 1.49\pi/3]$. Even though the pile is only 400 rows high, $P(f)$ clearly has an exponential tail.

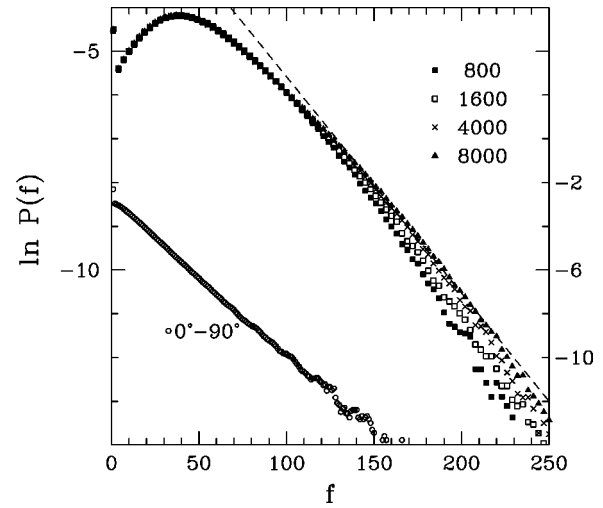The small f behavior of $P(f)$ depends on the range of

FIG. 4. Semilog plot of $P(f)$ vs f from the inner region of piles of varying heights; f is in arbitrary units and scaled with system size. The asymptote is roughly straight, and grows straighter with increasing system size. The inset (axis label on right) shows a similar plot with bond angles almost spanning $[0, \pi/2]$.

bond angles. However, in all cases one sees that $P(f \to 0) \neq 0$. This is in accordance with experiments [11], but *not* with the q-model. In the model in this paper, if a lattice site receives a large force from above, it transmits large forces to both its descendants. Since it is unlikely that either of its descendants gets a large force from its other ancestor, *both* descendants are usually destabilized outwards. The common descendant of the two destabilized grains thus receives no force. This yields a δ-function in $P(f)$ at $f = 0$ (actually, at the weight of a single grain), leading to $P(f \to 0) \neq 0$. Thus, the vector lattice model agrees with experiments in this respect because it has *arching*: weight applied at a point can propagate down outwards, completely avoiding the region directly below it. This is important in granular materials but is absent in the q-model.
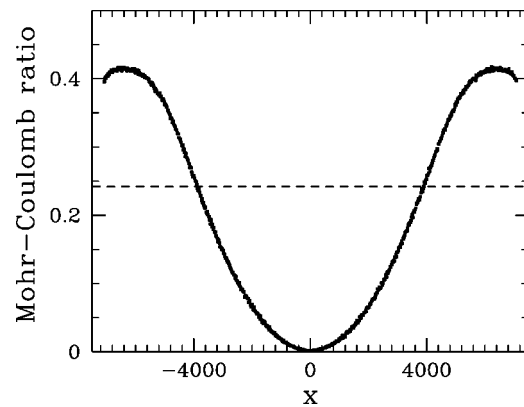

FIG. 5. Mohr Coulomb ratio across the bottom of the pile. The dashed line is $\sin^2\alpha$, where α is the angle of repose. The curve clearly goes above the line, showing that Eq. (2) is violated. (Where the curve crosses the line, relative to the boundary between the inner and outer regions, depends on $\theta_{\min,\max}$.) The pile height and units for x are as in Fig. 3.

One can also compute the correlation between vertical forces on sites (in the inner region of the pile) in the same row. Unlike the q-model this is not exactly zero, and depends on $[\theta_{min}, \theta_{max}]$, but is typically a few percent of the variance in f. This is consistent with the experimental observation [11] that vertical forces are uncorrelated.

Finally, Fig. 5 shows the MC ratio, the left-hand side of Eq. (2), obtained from the coarse grained stresses of Fig. 3. The dashed line shown is at a height of $\sin^2\alpha$, where α is the angle of repose (for a pile constructed as above, by pouring from a point). It is slightly tricky to define the angle of repose when a regular lattice is combined with random bond angles. If the width of the pile at a depth of m rows from the top is $w(m)$, and the large m limit of $m/w(m)$ is λ, since the average bond angle is $\pi/4$, α is taken to be $\tan^{-1}\lambda$. The curve goes above the straight line, violating the MC criterion.

In this paper, we have proposed and analyzed a two-dimensional lattice model with vector forces for granular materials. On an individual grain level, the model yields force chains, arching, and is consistent with an exponential tail to the distribution of vertical forces. Continuum stresses show an inner and outer region for a pile poured from a point, but the principal axes of the stress tensor are not fixed. Most importantly, the model violates the Mohr Coulomb criterion, calling in question its validity for poured granular heaps. It is not obvious how to extend this model to include inter-grain friction; off-lattice simulations using contact dynamics (with the heap in a container and prepared differently) have been done [16].

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